

# CARATHÉODORY THEOREMS FOR SLICE REGULAR FUNCTIONS

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**ABSTRACT.** In this paper a quaternionic sharp version of the Carathéodory theorem is established for slice regular functions with positive real part, which strengthens a weaken version recently established by D. Alpay et. al. using the Herglotz integral formula. Moreover, the restriction of positive real part can be relaxed so that the theorem becomes the quaternionic version of the Borel-Carathéodory theorem. It turns out that the two theorems are equivalent.

## 1. INTRODUCTION

The celebrated Carathéodory theorem for holomorphic functions with positive real part plays an important role in the geometric function theory of one complex variable (see [2, 7, 15]):

**Theorem 1.** *Let  $f : \mathbb{D} \rightarrow \mathbb{C}$  be a holomorphic function such that  $f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$  and  $\operatorname{Re} f(z) > 0$ , then for all  $z \in \mathbb{D}$ ,*

$$(1) \quad \frac{1 - |z|}{1 + |z|} \leq \operatorname{Re} f(z) \leq |f(z)| \leq \frac{1 + |z|}{1 - |z|};$$

and

$$(2) \quad |a_n| \leq 2, \quad \forall n \in \mathbb{N}.$$

Moreover, equality holds for the first or the third inequality in (1) at some point  $z_0 \neq 0$  if and only if  $f$  is of the form

$$f(z) = \frac{1 + e^{i\theta} z}{1 - e^{i\theta} z}, \quad \forall z \in \mathbb{C},$$

for some  $\theta \in \mathbb{R}$ .

A well-known Borel-Carathéodory theorem is a variant of the Carathéodory theorem relaxing the restriction of positive real part (For a Clifford-analytic version, see [16]):

**Theorem 2.** *Let  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  be a holomorphic function on  $\mathbb{D}$ , continuous up to the boundary  $\partial\mathbb{D}$ . Set  $A = \max_{|z|=1} \operatorname{Re} f(z)$ , then*

$$(3) \quad |a_n| \leq 2(A - \operatorname{Re} f(0)), \quad \forall n \in \mathbb{N};$$

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$$(4) \quad |f(z) - f(0)| \leq \frac{2r}{1-r}(A - \operatorname{Ref}(0)), \quad \forall |z| \leq r < 1.$$

It is quite natural to extend the results to higher dimensions or to other function classes. However, a great challenge may arise from extensions to non-holomorphic function classes, because of the failure of closeness under multiplication and composition.

The purpose of this article is to generalize the above two theorems to the setting of quaternions for slice regular functions. The theory of slice regular functions is initiated recently by Gentili and Struppa [12, 13]. The detailed up-to-date theory appears in the monographs [11, 6]. In particular, the slice regular product was introduced in the setting of quaternions in [9] for slice regular power series and in [3] for slice regular functions on symmetric slice domains. The geometric theory of slice regular functions of one quaternionic variable has been studied in [8, 18, 19]. Recently, the authors [17] establish the growth and distortion theorems for slice regular extensions of normalized univalent holomorphic functions with the help of a so-called convex combination identity.

Our main results in this article are the quaternionic versions of the Carathéodory theorem as well as the Borel-Carathéodory theorem for slice regular functions:

**Theorem 3.** *Let  $\mathbb{B}$  the open unit ball in the quaternions  $\mathbb{H}$ . If  $f : \mathbb{B} \rightarrow \mathbb{H}$  is a regular function for which  $f(q) = 1 + \sum_{n=1}^{\infty} q^n a_n$  and  $\operatorname{Ref}(q) > 0$ , then*

$$(5) \quad \frac{1 - |q|}{1 + |q|} \leq \operatorname{Ref}(q) \leq |f(q)| \leq \frac{1 + |q|}{1 - |q|}, \quad \forall q \in \mathbb{B},$$

and

$$(6) \quad |a_n| \leq 2, \quad \forall n \in \mathbb{N}.$$

Moreover, equality holds for the first or the third inequality in (5) at some point  $q_0 \neq 0$  if and only if  $f$  is of the form

$$f(q) = (1 - qe^{I\theta})^{-*} * (1 + qe^{I\theta}), \quad \forall q \in \mathbb{B},$$

for some  $\theta \in \mathbb{R}$  and some  $I \in \mathbb{H}$  with  $I^2 = -1$ .

Equality holds in (6) for some  $n_0 \in \mathbb{N}$ , i.e.  $|a_{n_0}| = 2$  if and only if

$$a_{kn_0} = 2 \left( \frac{a_{n_0}}{2} \right)^k, \quad \forall k \in \mathbb{N}.$$

In particular,  $|a_1| = 2$  if and only if

$$f(q) = (1 - qe^{I\theta_0})^{-*} * (1 + qe^{I\theta_0}), \quad \forall q \in \mathbb{B},$$

for some  $\theta_0 \in \mathbb{R}$  and some  $I \in \mathbb{H}$  with  $I^2 = -1$ .

**Theorem 4.** *Let  $f(q) = \sum_{n=0}^{\infty} q^n a_n$  be a slice regular function on  $\mathbb{B}$ . If  $A := \sup_{q \in \mathbb{B}} \operatorname{Ref}(q) < +\infty$ , then*

$$(7) \quad |a_n| \leq 2(A - \operatorname{Ref}(0)), \quad \forall n \in \mathbb{N};$$

$$(8) \quad |f(q) - f(0)| \leq \frac{2r}{1-r}(A - \operatorname{Ref}(0)), \quad \forall |q| \leq r < 1;$$

$$(9) \quad \operatorname{Re} f(q) \leq \frac{2r}{1+r}A + \frac{1-r}{1+r}\operatorname{Re} f(0), \quad \forall |q| \leq r < 1;$$

$$(10) \quad |f^{(n)}(q)| \leq \frac{2n!}{(1-r)^{n+1}}(A - \operatorname{Re} f(0)), \quad \forall |q| \leq r < 1, \quad n \in \mathbb{N}.$$

We remark as pointed to us by Sabadini that the weak inequalities

$$|\operatorname{Re}(a_n)| \leq 2$$

other than (6) can only be deduced with the approach as in [1] via the Herglotz integral formula, since  $d\mu_2(t)$  in Corollary 8.4 of [1] may not be a non-negative measure in general. Incidentally, the fact that  $|\operatorname{Re}(a_n)| \leq 2$  can also be proved by using the splitting lemma for slice regular functions and the Schwarz integral formula for holomorphic functions of one complex variable. However, the two methods can not be used to prove the sharp version. In this article we can apply the approach of finite average to deduce the strong version as in (6). In addition, a weakened inequality than (8) in Theorem 4 has been proved in [18] and our new approach allows to strengthen the statements. Finally, we point out that Theorems 3 and 4 turn out to be equivalent.

## 2. PRELIMINARIES

We recall in this section some preliminary definitions and results on slice regular functions. To have a more complete insight on the theory, we refer the reader to [11].

Let  $\mathbb{H}$  denote the non-commutative, associative, real algebra of quaternions with standard basis  $\{1, i, j, k\}$ , subject to the multiplication rules

$$i^2 = j^2 = k^2 = ijk = -1.$$

Every element  $q = x_0 + x_1i + x_2j + x_3k$  in  $\mathbb{H}$  is composed by the *real* part  $\operatorname{Re}(q) = x_0$  and the *imaginary* part  $\operatorname{Im}(q) = x_1i + x_2j + x_3k$ . The *conjugate* of  $q \in \mathbb{H}$  is then  $\bar{q} = \operatorname{Re}(q) - \operatorname{Im}(q)$  and its *modulus* is defined by  $|q|^2 = q\bar{q} = |\operatorname{Re}(q)|^2 + |\operatorname{Im}(q)|^2$ . We can therefore calculate the multiplicative inverse of each  $q \neq 0$  as  $q^{-1} = |q|^{-2}\bar{q}$ . Every  $q \in \mathbb{H}$  can be expressed as  $q = x + yI$ , where  $x, y \in \mathbb{R}$  and

$$I = \frac{\operatorname{Im}(q)}{|\operatorname{Im}(q)|}$$

if  $\operatorname{Im} q \neq 0$ , otherwise we take  $I$  arbitrarily such that  $I^2 = -1$ . Then  $I$  is an element of the unit 2-sphere of purely imaginary quaternions,

$$\mathbb{S} = \{q \in \mathbb{H} : q^2 = -1\}.$$

For every  $I \in \mathbb{S}$  we will denote by  $\mathbb{C}_I$  the plane  $\mathbb{R} \oplus I\mathbb{R}$ , isomorphic to  $\mathbb{C}$ , and, if  $\Omega \subseteq \mathbb{H}$ , by  $\Omega_I$  the intersection  $\Omega \cap \mathbb{C}_I$ . Also, for  $R > 0$ , we will denote the open ball centred at the origin with radius  $R$  by

$$B(0, R) = \{q \in \mathbb{H} : |q| < R\}.$$

We can now recall the definition of slice regularity.

**Definition 1.** Let  $\Omega$  be a domain in  $\mathbb{H}$ . A function  $f : \Omega \rightarrow \mathbb{H}$  is called *slice regular* if, for all  $I \in \mathbb{S}$ , its restriction  $f_I$  to  $\Omega_I$  is *holomorphic*, i.e., it has continuous partial derivatives and satisfies

$$\bar{\partial}_I f(x + yI) := \frac{1}{2} \left( \frac{\partial}{\partial x} + I \frac{\partial}{\partial y} \right) f_I(x + yI) = 0$$

for all  $x + yI \in \Omega_I$ .

As shown in [3], a class of domains, the so-called symmetric slice domains naturally qualify as domains of definition of slice regular functions.

**Definition 2.** Let  $\Omega$  be a domain in  $\mathbb{H}$ .  $\Omega$  is called a *slice domain* if  $\Omega$  intersects the real axis and  $\Omega_I$  is a domain of  $\mathbb{C}_I$  for any  $I \in \mathbb{S}$ .

Moreover, if  $x + yI \in \Omega$  implies  $x + y\mathbb{S} \subseteq \Omega$  for any  $x, y \in \mathbb{R}$  and  $I \in \mathbb{S}$ , then  $\Omega$  is called a *symmetric slice domain*.

From now on, we will omit the term ‘slice’ when referring to slice regular functions and will focus mainly on regular functions on  $B(0, R) = \{q \in \mathbb{H} : |q| < R\}$ . For regular functions the natural definition of derivative is given by the following (see [12, 13]).

**Definition 3.** Let  $f : B(0, R) \rightarrow \mathbb{H}$  be a regular function. The *slice derivative* of  $f$  at  $q = x + yI$  is defined by

$$\partial_I f(x + yI) := \frac{1}{2} \left( \frac{\partial}{\partial x} - I \frac{\partial}{\partial y} \right) f(x + yI).$$

Notice that the operators  $\partial_I$  and  $\bar{\partial}_I$  commute, and  $\partial_I f = \frac{\partial f}{\partial x}$  for regular functions. Therefore, the slice derivative of a regular function is still regular so that we can iterate the differentiation to obtain the  $n$ -th slice derivative

$$\partial_I^n f = \frac{\partial^n f}{\partial x^n}, \quad \forall n \in \mathbb{N}.$$

In what follows, for the sake of simplicity, we will direct denote the  $n$ -th slice derivative  $\partial_I^n f$  by  $f^{(n)}$  for every  $n \in \mathbb{N}$ .

As shown in [13], a quaternionic power series  $\sum_{n=0}^{\infty} q^n a_n$  with  $\{a_n\}_{n \in \mathbb{N}} \subset \mathbb{H}$  defines a regular function in its domain of convergence, which proves to be a open ball  $B(0, R)$  with  $R$  equal to the radius of convergence of the power series. The converse result is also true.

**Theorem 5. (Taylor Expansion)** *A function  $f$  is regular on  $B = B(0, R)$  if and only if  $f$  has a power series expansion*

$$f(q) = \sum_{n=0}^{\infty} q^n a_n \quad \text{with} \quad a_n = \frac{f^{(n)}(0)}{n!}.$$

A fundamental result in the theory of regular functions is described by the splitting lemma (see [13]), which relates slice regularity to classical holomorphy.

**Lemma 1. (Splitting Lemma)** *Let  $f$  be a regular function on  $B = B(0, R)$ , then for any  $I \in \mathbb{S}$  and any  $J \in \mathbb{S}$  with  $J \perp I$ , there exist two holomorphic functions  $F, G : B_I \rightarrow \mathbb{C}_I$  such that for every  $z = x + yI \in B_I$ , the following equality holds*

$$f_I(z) = F(z) + G(z)J.$$

The following version of the identity principle is one of the first consequences (as shown in [13]).

**Theorem 6. (Identity Principle)** *Let  $f$  be a regular function on  $B = B(0, R)$ . Denote by  $\mathcal{Z}_f$  the zero set of  $f$ ,*

$$\mathcal{Z}_f = \{q \in B : f(q) = 0\}.$$

If there exists an  $I \in \mathbb{S}$  such that  $B_I \cap \mathcal{Z}_f$  has an accumulation point in  $B_I$ , then  $f$  vanishes identically on  $B$ .

*Remark 1.* Let  $f_I$  be a holomorphic function on a disc  $B_I = B(0, R) \cap \mathbb{C}_I$  and let its power series expansion take the form

$$f_I(z) = \sum_{n=0}^{\infty} z^n a_n$$

with  $\{a_n\}_{n \in \mathbb{N}} \subset \mathbb{H}$ . Then the unique regular extension of  $f_I$  to the whole ball  $B(0, R)$  is the function defined by

$$f(q) := \text{ext}(f_I)(q) = \sum_{n=0}^{\infty} q^n a_n.$$

The uniqueness is guaranteed by the identity principle 6.

In general, the pointwise product of two regular functions is not a regular function. To guarantee the regularity of the product we need to introduce a new multiplication operation, the regular product (or  $*$ -product). On open balls centred at the origin, the  $*$ -product of two regular functions is defined by means of their power series expansions (see [9]).

**Definition 4.** Let  $f, g : B = B(0, R) \rightarrow \mathbb{H}$  be two regular functions and let

$$f(q) = \sum_{n=0}^{\infty} q^n a_n, \quad g(q) = \sum_{n=0}^{\infty} q^n b_n$$

be their series expansions. The regular product (or  $*$ -product) of  $f$  and  $g$  is the function defined by

$$f * g(q) = \sum_{n=0}^{\infty} q^n \left( \sum_{k=0}^n a_k b_{n-k} \right)$$

regular on  $B$ .

Notice that the  $*$ -product is associative and is not, in general, commutative. Its connection with the usual pointwise product is clarified by the following result (see [9]).

**Proposition 1.** Let  $f$  and  $g$  be two regular functions on  $B = B(0, R)$ . Then for all  $q \in B$ ,

$$f * g(q) = \begin{cases} f(q)g(f(q)^{-1}qf(q)) & \text{if } f(q) \neq 0; \\ 0 & \text{if } f(q) = 0. \end{cases}$$

We remark that if  $q = x + yI$  and  $f(q) \neq 0$ , then  $f(q)^{-1}qf(q)$  has the same modulus and same real part as  $q$ . Therefore  $f(q)^{-1}qf(q)$  lies in the same 2-sphere  $x + y\mathbb{S}$  as  $q$ . We obtain then that a zero  $x_0 + y_0I$  of the function  $g$  is not necessarily a zero of  $f * g$ , but an element on the same sphere  $x_0 + y_0\mathbb{S}$  does. In particular, a real zero of  $g$  is still a zero of  $f * g$ . To present a characterization of the structure of the zero set of a regular function  $f$ , we need to introduce the following functions.

**Definition 5.** Let  $f(q) = \sum_{n=0}^{\infty} q^n a_n$  be a regular function on  $B = B(0, R)$ . We define the *regular conjugate* of  $f$  as

$$f^c(q) = \sum_{n=0}^{\infty} q^n \bar{a}_n,$$

and the *symmetrization* of  $f$  as

$$f^s(q) = f * f^c(q) = f^c * f(q) = \sum_{n=0}^{\infty} q^n \left( \sum_{k=0}^n a_k \bar{a}_{n-k} \right).$$

Both  $f^c$  and  $f^s$  are regular functions on  $B$ .

We are now able to define the inverse element of a regular function  $f$  with respect to the  $*$ -product. Let  $\mathcal{Z}_{f^s}$  denote the zero set of the symmetrization  $f^s$  of  $f$ .

**Definition 6.** Let  $f$  be a regular function on  $B = B(0, R)$ . If  $f$  does not vanish identically, its *regular reciprocal* is the function defined by

$$f^{-*}(q) := f^s(q)^{-1} f^c(q)$$

regular on  $B \setminus \mathcal{Z}_{f^s}$ .

The following result shows that the regular product is nicely related to the point-wise quotient (see [20] and for general case see [21]).

**Proposition 2.** Let  $f$  and  $g$  be regular functions on  $B = B(0, R)$ . Then for all  $q \in B \setminus \mathcal{Z}_{f^s}$ ,

$$f^{-*} * g(q) = f(T_f(q))^{-1} g(T_f(q)),$$

where  $T_f : B \setminus \mathcal{Z}_{f^s} \rightarrow B \setminus \mathcal{Z}_{f^s}$  is defined by  $T_f(q) = f^c(q)^{-1} q f^c(q)$ . Furthermore,  $T_f$  and  $T_f^c$  are mutual inverses so that  $T_f$  is a diffeomorphism.

The following Schwarz lemma was proved in [13].

**Lemma 2. (Schwarz Lemma)** Let  $f : \mathbb{B} \rightarrow \mathbb{B}$  be a regular function. If  $f(0) = 0$ , then

$$|f(q)| \leq |q|$$

for all  $q \in \mathbb{B}$  and

$$|f'(0)| \leq 1.$$

Both inequalities are strict (except at  $q = 0$ ) unless  $f(q) = qu$  for some  $u \in \partial\mathbb{B}$ .

The following Leibniz rule was proved in [10].

**Proposition 3. (Leibniz rule)** Let  $f$  and  $g$  be regular functions on  $B = B(0, R)$ . Then

$$(f * g)' = f' * g + f * g'.$$

## 3. PROOF OF THE MAIN THEOREMS

In this section, we give the proofs of Theorems 3 and 4.

*Proof of Theorem 3.* Denote

$$g(q) = (f(q) - 1) * (f(q) + 1)^{-*}.$$

Notice that

$$(f(q) - 1) * (f(q) + 1)^{-*} = (f(q) + 1)^{-*} * (f(q) - 1),$$

we have

$$g(q) = (f(q) + 1)^{-*} * (f(q) - 1).$$

It is evident that  $g$  is regular on  $\mathbb{B}$  and  $g(0) = 0$ . By Proposition 2, we have

$$g(q) = (f \circ T_{1+f}(q) + 1)^{-1} (f \circ T_{1+f}(q) - 1).$$

This together with the fact that  $\operatorname{Re}(f(q)) > 0$  yields  $g(\mathbb{B}) \subseteq \mathbb{B}$ . Therefore, Lemma 2 implies that

$$(11) \quad |g(q)| \leq |q|, \quad \forall q \in \mathbb{B},$$

and

$$(12) \quad |g'(0)| \leq 1.$$

From the very definition of  $g$  and Proposition 2, it follows that

$$\begin{aligned} f(q) &= (1 - g(q))^{-*} * (g(q) + 1) \\ (13) \quad &= (1 - g \circ T_{1-g}(q))^{-1} (g \circ T_{1-g}(q) + 1). \end{aligned}$$

Recall that

$$|T_{1-g}(q)| = |q|, \quad \forall q \in \mathbb{B},$$

from (11) and (13) we obtain

$$(14) \quad |f(q)| \leq \frac{1 + |T_{1-g}(q)|}{1 - |T_{1-g}(q)|} = \frac{1 + |q|}{1 - |q|}$$

and similarly

$$(15) \quad \operatorname{Re}(f(q)) = \frac{1 - |g \circ T_{1-g}(q)|^2}{|1 - g \circ T_{1-g}(q)|^2} \geq \frac{1 - |q|^2}{(1 + |q|)^2} = \frac{1 - |q|}{1 + |q|}.$$

If equality holds in (14) or (15) at some point  $q_0 \neq 0$ , then it must be true that  $|g(q_0)| = |q_0|$ . Lemma 2 thus implies that

$$g(q) = qe^{I\theta}, \quad \forall q \in \mathbb{B},$$

for some  $I \in \mathbb{S}$  and some  $\theta \in \mathbb{R}$ , and hence

$$f(q) = (1 - qe^{I\theta})^{-*} * (1 + qe^{I\theta}), \quad \forall q \in \mathbb{B}.$$

The converse part can be easily verified by a simple calculation.

We now come to prove the assertion

$$|a_n| \leq 2, \quad n = 1, 2, \dots,$$

where  $a_n$  are the Taylor coefficients of  $f$  on the open unit ball  $\mathbb{B}$ .

First, from the very definition of  $g$  it follows that

$$f * (1 - g) = 1 + g,$$

from which as well as Proposition 3 we obtain that

$$f' * (1 - g) = (f + 1) * g'.$$

We now evaluate the preceding identity at  $q = 0$ . Since  $f(0) = 1$  and  $g(0) = 0$ , applying Proposition 1 we obtain

$$(16) \quad f'(0) = 2g'(0),$$

which together with (12) yields

$$(17) \quad |a_1| = |f'(0)| \leq 2.$$

If  $|a_1| = 2$ , i.e.,  $|g'(0)| = 1$ , it follows from Lemma 2 that

$$g(q) = qe^{I\theta}, \quad \forall q \in \mathbb{B},$$

for some  $I \in \mathbb{S}$  and some  $\theta \in \mathbb{R}$ , and hence

$$(18) \quad f(q) = (1 - qe^{I\theta})^{-*} * (1 + qe^{I\theta}), \quad \forall q \in \mathbb{B}.$$

Now we want to prove that  $|a_{n_0}| \leq 2$  for any fixed  $n_0 \geq 2$ . To this end, we need to construct a regular function  $\varphi$  with the same properties as  $f$ , whose Taylor coefficient of the first degree term is  $a_{n_0}$  so that we would conclude that  $|a_{n_0}| \leq 2$ .

Starting from the Taylor expansion of  $f$ , given by  $f(q) = 1 + \sum_{n=1}^{\infty} q^n a_n$ , we set

$$(19) \quad \varphi(q) = 1 + \sum_{m=1}^{\infty} q^m a_{mn_0} = 1 + qa_{n_0} + q^2 a_{2n_0} + \cdots$$

and

$$(20) \quad h(q) = \varphi(q^{n_0})$$

One can see from the radii of convergence of Taylor expansions that  $\varphi$  and  $h$  are slice regular functions on  $\mathbb{B}$ .

Now we claim that the restriction  $h_I$  of  $h$  to  $\mathbb{B}_I$  with  $I \in \mathbb{S}$  is exactly given by

$$(21) \quad h_I(z_I) = \frac{1}{n_0} \left( f_I(z_I) + f_I(z_I \omega_I) + \cdots + f_I(z_I \omega_I^{n_0-1}) \right), \quad \forall z_I \in \mathbb{B}_I,$$

where  $f_I$  is the restriction of  $f$  to  $\mathbb{B}_I$  and  $\omega_I \in \mathbb{C}_I$  is any quaternionic primitive  $n_0$ -th root of unity. From (20) the claim is equivalent to the identity

$$(22) \quad \frac{1}{n_0} \sum_{k=0}^{n_0-1} f_I(z_I \omega_I^k) = 1 + \sum_{m=1}^{\infty} q^{mn_0} a_{mn_0}.$$

To prove this, we apply the power series expansion of  $f$  and obtain

$$(23) \quad \begin{aligned} \frac{1}{n_0} \sum_{k=0}^{n_0-1} f_I(z_I \omega_I^k) &= 1 + \frac{1}{n_0} \left( \sum_{m=1}^{\infty} z_I^m a_m + \sum_{m=1}^{\infty} z_I^m \omega_I^m a_m + \cdots + \sum_{m=1}^{\infty} z_I^m \omega_I^{(n_0-1)m} a_m \right) \\ &= 1 + \frac{1}{n_0} \sum_{m=1}^{\infty} z_I^m \left( \sum_{k=0}^{n_0-1} \omega_I^{km} \right) a_m. \end{aligned}$$

Since  $\omega_I$  is a primitive  $n$ -th root of unity in the plane  $\mathbb{C}_I$ , we have

$$(24) \quad \frac{1}{n_0} \sum_{k=0}^{n_0-1} \omega_I^{km} = \begin{cases} 1, & n_0 \mid m; \\ 0, & \text{otherwise.} \end{cases}$$



Indeed, if  $n_0 \mid m$ , then  $\omega_I^m = 1$  such that each summand in (24) equals 1, so is the average. Otherwise, then

$$\sum_{k=0}^{n_0-1} \omega_I^{km} = \frac{1 - \omega_I^{n_0 m}}{1 - \omega_I^m} = 0$$

and (24) holds true. Inserting (24) into (23), we get (22) and this proves the claim.

We want to show that

$$(25) \quad \operatorname{Re}(\varphi(q)) > 0$$

for any  $q \in \mathbb{B}$ . Let  $q \in \mathbb{B}$  be given and take  $u \in \mathbb{B}$  such that  $u^{n_0} = q$ . Hence due to (20) we have

$$\varphi(q) = h(u).$$

To prove (25), It suffices to prove that

$$\operatorname{Re}(h(u)) > 0, \quad \forall u \in \mathbb{B}.$$

Thus we only need to show the result for the restrictions of  $h$  to any complex plane  $h_I$ , i.e.,

$$\operatorname{Re}(h_I(z_I)) > 0, \quad \forall z_I \in B_I, \quad \forall I \in \mathbb{S}.$$

This follows easily from (21) and the assumption that  $\operatorname{Re}(f(q)) > 0$ .

Therefore, the regular function  $\varphi$  we have constructed via  $f$  has the same properties as  $f$ , whose Taylor coefficient of the first degree term is  $a_{n_0}$ . Consequently, we obtain that  $|a_{n_0}| \leq 2$  for any  $n_0 \in \mathbb{N}$ .

Finally, if there exists  $n_0 \in \mathbb{N}$  such that  $|a_{n_0}| = 2$ , then the argument similar to the extremal case for  $n = 1$  implies that

$$\varphi(q) = (1 - qe^{I\theta_0})^{-*} * (1 + qe^{I\theta_0}) = 1 + 2 \sum_{m=1}^{\infty} q^m e^{Im\theta_0}, \quad \forall q \in \mathbb{B},$$

for some  $I \in \mathbb{S}$  and some  $\theta_I \in \mathbb{R}$ , which together with (26) implies that

$$a_n = 2 \left( \frac{a_{n_0}}{2} \right)^{\frac{n}{n_0}} \quad \text{if} \quad n_0 \mid n,$$

where  $a_{n_0} = 2e^{In_0\theta_0}$ . In particular, when  $n_0 = 1$ , this is clearly equivalent to (18). Now the proof is completed.  $\square$

To prove Theorem 4, we need a useful lemma, which have been proved in [14, Theorem 2.7]. Here we provide an alternative proof by applying the open mapping theorem.

**Lemma 3.** *Let  $f : B(0, R) \rightarrow \mathbb{H}$  be a regular function. If  $\operatorname{Re} f$  attains its maximum at some point  $q_0$ , then  $f$  is constant in  $B(0, R)$ .*

*Proof.* We argue by contradiction, and suppose that  $f$  is not constant while  $\operatorname{Re} f$  attains its maximum at some point  $q_0 = x_0 + y_0 I_0$ . The real part  $\operatorname{Re}(f(x + yI_0))$  of the restriction  $f_{I_0}$  of  $f$  to  $B_{I_0}(0, R)$  is a harmonic function of two variables  $x, y$ . Thus  $\operatorname{Re} f_{I_0}$  is constant on  $B_{I_0}(0, R)$ . In particular,  $\operatorname{Re}(f_{I_0})$  attains its maximum at the point  $x_0 \in (-R, R)$ , which belongs to  $B_I(0, R)$  for all  $I \in \mathbb{S}$ . As a consequence,  $\operatorname{Re} f$  is constant on  $B_I(0, R)$  for all  $I \in \mathbb{S}$  and hence on  $B(0, R)$ . On the contrary, from our assumption that  $f$  is not constant it follows that  $f$  is not constant either, in view of the open mapping theorem. The contradiction concludes the proof.  $\square$

*Remark 2.* The preceding lemma can also be proved by using the splitting lemma for slice regular functions and the maximum modulus principle for harmonic functions.

*Proof of Theorem 4.* The proof of inequality (8) is similar to the one given in [18], but simpler. The result is obvious if  $f$  is constant. Otherwise, its real part  $\text{Re}f$  is not constant either, in view of the open mapping theorem, then it follows from Lemma 3 that

$$\text{Re}f(q) < A, \quad \forall q \in \mathbb{B}.$$

First, we assume that  $f(0) = 0$ . Consider the function

$$g(q) = (2A - f(q))^{-*} * f(q) = f(q) * (2A - f(q))^{-*},$$

which is regular on  $\mathbb{B}$  and  $|g(q)| < 1$ . The last assertion follows from the facts that  $g = (2A - f \circ T_{2A-f})^{-1} f \circ T_{2A-f}$  and  $\text{Re}f < A$ . Since  $g(0) = 0$ , it follows from Theorem 2 that

$$(26) \quad |g(q)| \leq |q|, \quad \forall q \in \mathbb{B},$$

and

$$(27) \quad |g'(0)| \leq 1.$$

From the very definition of  $g$  and Proposition 2 it follows that

$$f(q) = 2Ag(q) * (g(q) - 1)^{-*} = 2A(g(q) - 1)^{-*} * g(q) = 2A(g \circ T_{g^{-1}}(q) - 1)^{-1} g \circ T_{g^{-1}}(q),$$

and hence

$$(28) \quad |f(q)| \leq 2A \frac{|T_{g^{-1}}(q)|}{1 - |T_{g^{-1}}(q)|} = 2A \frac{|q|}{1 - |q|}, \quad \forall q \in \mathbb{B}.$$

Moreover,

$$(29) \quad \begin{aligned} \text{Re}f(q) &= 2A \frac{|g \circ T_{g^{-1}}(q)|^2 - \text{Re}(g \circ T_{g^{-1}}(q))}{|1 - g \circ T_{g^{-1}}(q)|^2} \\ &= A \left( 1 - \frac{1 - |g \circ T_{g^{-1}}(q)|^2}{1 + |g \circ T_{g^{-1}}(q)|^2 - 2\text{Re}(g \circ T_{g^{-1}}(q))} \right) \\ &\leq A \left( 1 - \frac{1 - |g \circ T_{g^{-1}}(q)|^2}{(1 + |g \circ T_{g^{-1}}(q)|)^2} \right) \\ &= 2A \frac{|g \circ T_{g^{-1}}(q)|}{1 + |g \circ T_{g^{-1}}(q)|} \\ &= 2A \left( 1 - \frac{1}{1 + |g \circ T_{g^{-1}}(q)|} \right) \\ &= 2A \left( 1 - \frac{1}{1 + |q|} \right) \\ &= 2A \frac{|q|}{1 + |q|}, \end{aligned}$$

since  $|g \circ T_{g^{-1}}(q)| \leq |T_{g^{-1}}(q)| = |q|$ , which follows from (26).

Again from the very definition of  $g$ , we obtain that

$$(2A - f) * g = f,$$

which together with Proposition 3 implies that

$$f' * (1 + g) = (2A - f) * g'.$$

We now evaluate the preceding identity at  $q = 0$ . Since  $f(0) = 0$  and  $g(0) = 0$ , applying Proposition 1 we obtain

$$(30) \quad f'(0) = 2Ag'(0),$$

which together with (27) yields

$$(31) \quad |f'(0)| \leq 2A.$$

For general case, we consider the function  $f - f(0)$ , replacing  $f$ ,  $A$  by  $f - f(0)$  and  $A - \operatorname{Ref}(0)$  in inequalities (28), (29) and (31) respectively, yields that

$$(32) \quad |f(q) - f(0)| \leq 2(A - \operatorname{Ref}(0)) \frac{|q|}{1 - |q|}, \quad \forall q \in \mathbb{B},$$

(33)

$$\operatorname{Ref}(q) \leq \operatorname{Ref}(0) + \frac{2|q|}{1 + |q|} (A - \operatorname{Ref}(0)) \leq \frac{2|q|}{1 + |q|} A + \frac{1 - |q|}{1 + |q|} \operatorname{Ref}(0), \quad \forall q \in \mathbb{B},$$

and

$$(34) \quad |a_1| = |f'(0)| = |(f - f(0))'(0)| \leq 2(A - \operatorname{Ref}(0)).$$

Now inequality (8) follows from the maximum modulus principle and (32). Similarly, inequality (9) follows from Lemma 3 and (33).

Next, we want to prove that  $|a_{n_0}| \leq 2(A - \operatorname{Ref}(0))$  for any fixed  $n_0 \geq 2$ . The proof is similar to the one given in Theorem 3. To this end, we need to construct a regular function  $\varphi$  with the same properties as  $f$ , whose Taylor coefficient of the first degree term is  $a_{n_0}$  so that we would conclude that  $|a_{n_0}| \leq 2(A - \operatorname{Ref}(0))$ .

Starting from the Taylor expansion of  $f$ , given by  $f(q) = \sum_{n=0}^{\infty} q^n a_n$ , we set

$$(35) \quad \varphi(q) = \sum_{m=0}^{\infty} q^m a_{mn_0} = a_0 + qa_{n_0} + q^2 a_{2n_0} + \cdots,$$

and

$$(36) \quad h(q) = \varphi(q^{n_0}).$$

One can see from the radii of convergence of Taylor expansions that  $\varphi$  and  $h$  are slice regular functions on  $\mathbb{B}$ , respectively.

Reasoning as in the proof of Theorem 3 gives that the restriction  $h_I$  of  $h$  to  $\mathbb{B}_I$  with  $I \in \mathbb{S}$  is exactly given by

$$(37) \quad h_I(z_I) = \frac{1}{n_0} \left( f_I(z_I) + f_I(z_I \omega_I) + \cdots + f_I(z_I \omega_I^{n_0-1}) \right), \quad \forall z_I \in \mathbb{B}_I,$$

where  $f_I$  is the restriction of  $f$  to  $\mathbb{B}_I$  and  $\omega_I \in \mathbb{C}_I$  is any quaternionic primitive  $n_0$ -th root of unity. It follows from (37) that

$$A_h := \sup_{q \in \mathbb{B}} \operatorname{Re}(h(q)) \leq A_f = A,$$

and hence

$$A_\varphi := \sup_{q \in \mathbb{B}} \operatorname{Re}(\varphi(q)) \leq A_f = A$$

by the relation (36). Moreover, we have  $\varphi(0) = a_0 = f(0)$ .

As a result, the regular function  $\varphi$  constructed via  $f$  has the same properties as  $f$ , whose Taylor coefficient of the first degree term is  $a_{n_0}$ . Consequently, we obtain that  $|a_{n_0}| = |\varphi'(0)| \leq 2(A_\varphi - \operatorname{Ref}(0)) \leq 2(A - \operatorname{Ref}(0))$  for any  $n_0 \in \mathbb{N}$ .

Finally, we come to prove inequality (10). It follows from the Taylor expansion of  $f$  that

$$f^{(n)}(q) = \sum_{m=n}^{\infty} m(m-1) \cdots (m-n+1) q^{m-n} a_m,$$

and hence by (7),

$$\begin{aligned} |f^{(n)}(q)| &\leq \sum_{m=n}^{\infty} m(m-1) \cdots (m-n+1) |a_m| |q|^{m-n} \\ &\leq 2(A - \operatorname{Ref}(0)) \sum_{m=n}^{\infty} m(m-1) \cdots (m-n+1) r^{m-n} \\ (38) \quad &= 2(A - \operatorname{Ref}(0)) \left( \frac{d^n}{dt^n} \sum_{m=0}^{\infty} t^m \right) \Big|_{t=r} \\ &= \frac{2n!}{(1-r)^{n+1}} (A - \operatorname{Ref}(0)) \end{aligned}$$

for all  $|q| \leq r < 1$  and  $n \in \mathbb{N}$ . Incidentally, inequality (8) can also be proved in the same manner as above. Now the proof is completed.  $\square$

Finally we come to show the equivalence of Theorems 3 and 4.

**Theorem 7.** *Theorems 3 and 4 are equivalent.*

*Proof.* Set  $A = 0$  and consider the function  $-f$ , then (5) and (6) follow from Theorem 4.

Now we deduce Theorem 4 from Theorem 3. Let  $f$  be described in Theorem 4 and be not constant, then

$$\operatorname{Ref}(q) < A, \quad \forall q \in \mathbb{B}.$$

Consider the function

$$g(q) = \frac{(A + \operatorname{Ref}(0)) - (f(q) + \overline{f(0)})}{A - \operatorname{Ref}(0)}, \quad \forall q \in \mathbb{B}.$$

Then  $g$  is regular on  $\mathbb{B}$ ,  $g(0) = 1$  and

$$\operatorname{Reg}(q) = \frac{A - \operatorname{Ref}(q)}{A - \operatorname{Ref}(0)} > 0,$$

i.e.  $g$  satisfies the assumptions given in Theorem 3. Therefore,

$$|a_n| = \left| \frac{f^{(n)}(0)}{n!} \right| = \left| \frac{g^{(n)}(0)}{n!} \right| (A - \operatorname{Ref}(0)) \leq 2(A - \operatorname{Ref}(0)),$$

and

$$\frac{A - \operatorname{Ref}(q)}{A - \operatorname{Ref}(0)} = \operatorname{Reg}(q) > \frac{1 - |q|}{1 + |q|},$$

from which it follows that

$$\operatorname{Ref}(q) \leq \frac{2|q|}{1 + |q|} A + \frac{1 - |q|}{1 + |q|} \operatorname{Ref}(0), \quad \forall q \in \mathbb{B}.$$

Now inequality (9) follows from Lemma 3. Inequalities (8) and (10) follow from (7) as in the proof of Theorem 4. This completes the proof.  $\square$

## REFERENCES

1. Alpay D., Colombo F., Kimsey D. P., Sabadini I.: An extension of Herglotz's theorem to the quaternions. arXiv preprint, arXiv:1403.0079, 2014-arxiv.org.
2. Carathéodory C.: Über den Variabilitätsbereich der Fourierschen Konstanten von positiven harmonischen Funktionen. Rend. Circ. Mat. Palermo. 32, 193-217 (1911)
3. Colombo F., Gentili G., Sabadini I., Struppa D. C.: Extension results for slice regular functions of a quaternionic variable. Adv. Math. 222(5), 1793-1808 (2009)
4. Colombo F., Sabadini I.: A structure formula for slice monogenic functions and some of its consequences, in Hypercomplex Analysis, Trends in Mathematics (Birkhäuser, Basel, 2009), pp. 101-114
5. Colombo F., Sabadini I., Struppa D. C.: An extension theorem for slice monogenic functions and some of its consequences. Israel J. Math. 177, 369-389 (2010)
6. Colombo F., Sabadini I., Struppa D. C.: Noncommutative functional calculus. Theory and applications of slice hyperholomorphic functions. Progress in Mathematics, vol. 289. Birkhäuser/Springer, Basel (2011)
7. Duren P. L.: Univalent functions. Grundlehren der Mathematischen Wissenschaften, 259, Springer-Verlag, New York (1983)
8. Gentili G., Sarfatti G.: Landau-Toeplitz theorems for slice regular functions over quaternions. Pacific J. Math. 265(2), 381-404 (2013)
9. Gentili G., Stoppato C.: Zeros of regular functions and polynomials of a quaternionic variable. Mich. Math. J. 56(3), 655-667 (2008)
10. Gentili G., Stoppato C.: Power series and analyticity over the quaternions. Math. Ann. 352(1), 113-131 (2012)
11. Gentili G., Stoppato C., Struppa D. C.: Regular functions of a quaternionic variable. Springer Monographs in Mathematics, Springer, Berlin-Heidelberg (2013)
12. Gentili G., Struppa D. C.: A new approach to Cullen-regular functions of a quaternionic variable. C. R. Math. Acad. Sci. Paris. 342(10), 741-744 (2006)
13. Gentili G., Struppa D. C.: A new theory of regular functions of a quaternionic variable. Adv. Math. 216(1), 279-301 (2007)
14. Gentili G., Struppa D. C.: On the real part of slice regular functions. Preprint (2011)
15. Graham I., Kohr G.: Geometric function theory in one and higher dimensions. Monographs and Textbooks in Pure and Applied Mathematics, 255. Marcel Dekker, Inc., New York (2003)
16. Göllebeck K., Morais J., Cerejeiras P.: Borel-Carathéodory type theorem for monogenic functions. Complex Anal. Oper. Theory. 3(1), 99-112 (2009)
17. Ren G. B., Wang X. P.: The growth and distortion theorems for slice regular functions. submitted.
18. Rocchetta C. D., Gentili G., Sarfatti G.: The Bohr theorem for slice regular functions. Math. Nachr. 285(17-18), 2093-2105 (2012)
19. Rocchetta C. D., Gentili G., Sarfatti G.: A Bloch-Landau theorem for slice regular functions. Advances in hypercomplex analysis. Springer INdAM Ser (Springer, Milan, 2013), pp. 55-74
20. Stoppato C.: Poles of regular quaternionic functions. Complex Var. Elliptic Equ. 54(11), 1001-1018 (2009)
21. Stoppato C.: Singularities of slice regular functions. Math. Nachr. 285(10), 1274-1293 (2012)

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